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THE BOGGESS–POLKING EXTENSION THEOREM FOR CR FUNCTIONS ON MANIFOLDS WITH CORNERS

BY

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ABSTRACT

We give the "boundary version" of the Boggess–Polking CR extension theorem. Let M and N be real generic submanifolds of $\mathbb{C}^n$ with $N \subset M$ and let V be a "wedge" in M with "edge" N and "profile" $\Sigma \subset T_N M$ in a neighborhood of a point z_o. We identify in natural manner

$$T_N M \overset{k}{\hookrightarrow} \frac{T^{\mathbb{C}}M|_N}{T^{\mathbb{C}}N}, \frac{TM}{T^{\mathbb{C}}M} \underset{j}{\overset{\sim}{\longrightarrow}} T_M X$$

and assume that for a holomorphic vector field L tangent to M and verifying $L(z_o) + \bar{L}(z_o) \in k(\Sigma_{z_o})$ we have that the Levi form $j\mathcal{L}(L)_{z_o} := j\left(\frac{1}{2i}[L, \bar{L}]_{z_o}\right)$ takes a value $iv_o \in T_M X_{z_o}$, $iv_o \neq 0$ (say $|v_o| = 1$). Then we prove that CR functions on V extend $\forall\varepsilon$ to a wedge V_1 "attached" to V in direction of a vector field $i\mathcal{V}$ such that $|\mathrm{pr}(i\mathcal{V}(z_o)) - iv_o| < \varepsilon$ (where pr is the projection pr: $T_N X \to T_M X|_N$). We then prove that when the Levi cone "relative to Σ" $iZ_\Sigma =$ convex hull $\{j\mathcal{L}(L)_{z_o} | L(z_o) + \bar{L}(z_o) \in k(\Sigma)_{z_o}\}$ is open in $T_M X$, then CR functions extend to a "full" wedge with edge N (that is, with a profile which is an open cone of $T_N X$). Finally, we prove that if f is defined in a couple of wedges $\pm V$ with profiles $\pm\Sigma$ such that $iZ_\Sigma = T_M X$, and is continuous up to N, then f is in fact holomorphic at z_o.

Let X be a complex manifold of dimension n and let M be a C^5 generic CR manifold of X, N a generic C^5 submanifold of M, and V a wedge of M with edge N. This is by definition the diffeomorphic image of a straight wedge of $\mathbb{R}^{2n}$ that can be described as follows. In a coordinate system in a neighborhood of

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$z_o = 0$, we write $X \simeq \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{l-m} \times \mathbb{C}^{n-l}$ with coordinates (z', z'', w) and also use the notations $z = (z', z'')$ and $z = x + iy$. We introduce a new parameter $t \in \mathbb{R}^{l-m}$ and an open convex cone $\Gamma \subset \mathbb{R}^{l-m}$. For a convenient vector function $h = (h_i)_{i=1,\dots,l}$ of class C^5 in the variables $(x, t, w) \in \mathbb{R}^l \times \mathbb{R}^{l-m} \times \mathbb{C}^{n-l}$ we have

$$(1) \qquad \begin{cases} V\colon y = h(x,t,w) & \text{for } t \in \Gamma, \\ M\colon y = h(x,t,w) & \text{for } t \in \mathbb{R}^{l-m}, \\ N\colon y = h(x,0,w). \end{cases}$$

Moreover, we can suppose that under our choice of coordinates we have

$$(2) \qquad \partial_x h(z_o) = \partial_w h(z_o) = \partial_{\bar{w}} h(z_o) = 0,$$

and

$$(3) \qquad \partial_{t_j} h_i(z_o) = \begin{cases} \delta_{ij} & \text{for } m+1 \le i, j \le l, \\ 0 & \text{for } i \le m. \end{cases}$$

Hence if we define $F\colon \mathbb{R}^l \times \mathbb{R}^{l-m} \times \mathbb{C}^{n-l} \to \mathbb{C}^n$ by $F(x,t,w) := (x + \mathrm{i}h(x,t,w), w)$, then F has (real) rank $2n - m$ and we obtain

$$(4) \quad M = F(\mathbb{R}^l \times \mathbb{R}^{l-m} \times \mathbb{C}^{n-l}),\ V = F(\mathbb{R}^l \times \Gamma \times \mathbb{C}^{n-l}),\ N = F(\mathbb{R}^l \times \{0\} \times \mathbb{C}^{n-l}).$$

Remark: Note that by (3) and by the Implicit Function Theorem, we can solve with respect to t the last $l - m$ equations $y_i = h_i(x,t,w)$ $i = m+1, \dots, l$ and get

$$(5) \qquad t = t(x, y'', w).$$

We plug (5) in the first m equations of M and define

$$(6) \qquad g_i(x, y'', w) := \begin{cases} h_i(x, t(x,y'',w), w) & \text{for } i = 1, \dots, m, \\ h_i(x, 0, w) & \text{for } i = m+1, \dots, l. \end{cases}$$

This yields

$$(7) \qquad \begin{cases} M = \{y_i = g_i(x,y'',w)\ i = 1,\dots,m\}, \\ N = \{y_i = g_i(x,y'',w)\ i = 1,\dots,l\}, \\ V = \{y_i = g_i(x,y'',w)\ i = 1,\dots,m,\ y'' \in \Sigma_{x,w}\}, \end{cases}$$

where $\Sigma_{x,w}$ is the wedge in the y''-plane with vertex $\{y'' = (g_i(x,0,w))_{i=m+1,\dots,l}\}$ defined by $\Sigma_{x,w} = (g_i(x,\Gamma,w))_i$. Thus we have now a wedge condition "$y'' \in \Sigma_{x,w}$" instead of a simpler cone condition "$t \in \Gamma$". This is why the adjunction of the parameter t is convenient for the purpose of the present paper and we will use (1) rather than (7).

We shall denote by T (resp. $T^{\mathbb{C}}$) the tangent (resp. the complex tangent) bundle. Note that $T_{z_o}N = \mathbb{R}^m_{x'} \times \mathbb{R}^{l-m}_{x''} \times \mathbb{C}^{n-l}_w$, $T_{z_o}M \simeq \mathbb{R}^m_{x'} \times \mathbb{C}^{l-m}_{z''} \times \mathbb{C}^{n-l}_w$, $T_{z_o}V \simeq \mathbb{R}^m_{x'} \times (\mathbb{R}^{l-m}_{x''} + i\Gamma_{y''}) \times \mathbb{C}^{n-l}$.

By the genericity of N we have an identification

$$\frac{iTN \cap TM}{T^{\mathbb{C}}N} \to T_N M \tag{8}$$

given by $[ia]$ (modulo $T^{\mathbb{C}}N$) $\mapsto [ia]$ (modulo TN). For z close to z_o we also have an identification

$$T_N M_z \to \mathbb{R}^{l-m} \tag{9}$$

given by

$$[ia] \mapsto (\Re\mathrm{e} < \partial r_j(z_o), ia >)_{j=m+1,\dots,l} \tag{10}$$

(where we have used the notation $r_j = y_j - h_j$).

In particular, if we evaluate at z_o, (10) is induced by the projection $(z', z'', w) \mapsto y''$ (because the equations r_j are normalized at z_o). Hence the inverse

$$\mathbb{R}^{l-m} \to \left(\frac{iTN \cap TM}{T^{\mathbb{C}}N}\right)_{z_o}$$

of the composition of (8) and (9) is given by $b \mapsto (0, ib, 0)$, where 0 stands for the z' and w entries. (We shall often write ib instead of $(0, ib, 0)$.) In particular, this induces first an embedding $\Gamma \hookrightarrow (T_N M)_{z_o}$ and then also

$$\Gamma \hookrightarrow \left(\frac{iTN \cap TM}{T^{\mathbb{C}}N}\right)_{z_o} \hookrightarrow \left(\frac{T^{\mathbb{C}}M}{T^{\mathbb{C}}N}\right)_{z_o}.$$

Proposition 1: (i) *We can choose coordinates such that M, V and N can be represented as in (7) and, moreover,*

$$g_i(x, y'', w) = \bar{\partial}_z \partial_z g_i(0)(\overline{(z'', w)}, (z'', w)) + o(x', z'', w)^2 \quad \forall i \le m.$$

(ii) *Let $b \in \mathbb{R}^{l-m} \setminus \{0\}$, $c \in \mathbb{C}^{n-l}$. Then we can arrange that, under a choice of coordinates, all the above conditions are fullfilled and, moreover, $ib + c$ is either ie_{m+1} or $ie_{m+1} + e_{l+1}$, where e_j is the j-th unit vector.*

Proof: (i) It is a classical result on the normal form of a CR manifold (cf. [4] p. 109) that for a transformation of type

$$\begin{cases} \tilde{z}' = z' - P_2(z', z'', w) \ (P_2 \text{ a polynomial of degree 2}) \\ \tilde{z}'' = z'' \\ \tilde{w} = w \end{cases}$$

we can give the required form to g without destroying the properties of h.

(ii) This is obvious by means of a linear change of coordinates in $\mathbb{R}^{l-m}_{y''}$ and $\mathbb{C}^{n-l}_w$. ∎

Using the identification $\Gamma \hookrightarrow (T_N M)_{z_o}$, $b \mapsto [ib]$ we shall think of $i\Gamma$ as the "supplementary tangent directions of V with respect to N". In the terminology of [8] we shall say that V is "attached" to N at z_o in the directions of $i\Gamma$. (Thus M will be attached in all directions of $i\mathbb{R}^{l-m}$.)

We shall also use the identifications $T_N X \simeq \mathbb{R}^l$ and $T_M X \mapsto \mathbb{R}^m$ given by $[ia] \mapsto (\Re\mathrm{e} < \partial r_j, ia >)_{1\le j\le l}$ and $[ia] \mapsto (\Re\mathrm{e} < \partial r_j, ia >)_{1\le j\le m}$ respectively. We shall also denote by pr the projection $\mathbb{R}^l \to \mathbb{R}^m$.

THEOREM 2: *Assume N, M, V of class C^5 and $(\bar\partial\partial r_j(z_o)(\overline{ib+c}, ib+c))_{j=1,\dots,m} \neq 0$ for $b \in \Gamma \cup \{0\}$, $c \in \mathbb{C}^{n-l}$. We denote by v_o the above vector and suppose without loss of generality $|v_o| = 1$. Then $\forall\varepsilon$ there exists a vector field $\mathcal{V}(z) \in \mathbb{R}^l$, $z \in V \cup N$, with $|\mathrm{pr}(\mathcal{V}(z_o)) - v_o| < \varepsilon$, and a wedge V_1 of class $C^{3,\alpha}$ attached to V in direction $i\mathcal{V}$ such that any CR function on V extends to be CR on V_1.*

The proof is given after all statements. When M is a hypersurface and N is totally real, the above statement is equivalent to [6, Th. 1.4].

Important Remark: Theorem 2 has in fact an intrinsic statement. First we can define intrinsically the "profile" of V by

$$\Sigma|_N = \frac{T(V \cup N)|_N}{TN},$$

an open cone of $T_N M$. If V is parametrized as in (1), then we shall have $\Sigma_{z_o} = i\Gamma$. (We shall then say that V is attached to N in the directions of Σ.) Also, the Levi form can be intrinsically defined by $\frac{1}{2i}[L, \bar L]$ (modulo $T^{\mathbb{C}}M$) for all $L \in T^{1,0}M$. This is a real form which takes values in $\frac{TM}{T^{\mathbb{C}}M}$. Using the identification

$$\frac{TM}{T^{\mathbb{C}}M} \overset{j}{\simeq} T_M X$$

(induced by the multiplication by i) we shall rather consider $j(\frac{1}{2i}[L, \bar L])$ (modulo TM) in $T_M X$.

Thus assume that for $[ib] \in \Sigma_{z_o} \subset T_N M$ identified to $[ib] \in \frac{T^{\mathbb{C}}M}{T^{\mathbb{C}}N}$, and for a vector field $L \in T^{1,0}M$ with $L(z_o) + \bar L(z_o) = ib + c$, $c \in T^{\mathbb{C}}N$, we have $j(\frac{1}{2i}[L, \bar L]) \neq 0$ in $T_M X$. Then all CR functions on V will extend in the direction of a vector field $i\mathcal{V}$ with $|\mathrm{pr}(i\mathcal{V}(z_o)) - j(\frac{1}{2i}[L, \bar L](z_o)| < \varepsilon$, pr being the projection $T_N X \to T_M X|_N$).

We give now two corollaries but before we need the following

Definition 3: For $\Gamma \subset \mathbb{R}^{l-m}_{y''}$ we define

$$Z_\Gamma := \text{c.h.}(\bar\partial\partial r_j(z_o)(\overline{ib+c}, ib+c))_{j=1,\dots,m} \quad \forall b \in \Gamma,\ \forall c \in \mathbb{C}^{n-l},$$

where c.h. stands for convex hull.

Then $\Gamma + Z_\Gamma$ will be a cone of $\mathbb{R}^{l-m} \times \mathbb{R}^m$ (open if Z_Γ is open in $\mathbb{R}^m$).

COROLLARY 4: *Assume Z_Γ is open in $\mathbb{R}^m$. Then for all $Z' \subset\subset (Z_\Gamma)_{z_o}$ there exists an open cone $\tilde{Z}' \subset \mathbb{R}^l$ whose projection in $\mathbb{R}^m$ is Z', such that any CR function on V extends holomorphically to $((V \cap B) + i\tilde{Z}') \cap B$ for a suitable neighborhood B of z_o.*

The proof will be given at the end of the paper. We consider now the antipodal cone $-\Gamma$ to Γ and let $\pm V$ be defined by $y = h(x, t, w)$ for $t \in \pm\Gamma$.

COROLLARY 5: *Assume $Z_\Gamma = \mathbb{R}^m$. Then any CR function f on $V^+ \cup V^-$, continuous up to N, extends holomorphically to a full neighborhood of z_o.*

Proof: We know that f extends to

$$\Big(((V \cap B) + i\tilde{Z}_1) \cap B\Big) \cup \Big(((V^- \cap B) + i\tilde{Z}_2) \cap B\Big)$$

where $\tilde{Z}_i, = 1, 2$ are cones of $\mathbb{R}^l$ whose projection in $\mathbb{R}^m$ is the full $\mathbb{R}^m$. We have

$$\begin{aligned}&\Big(((V^+ \cap B) + i\tilde{Z}_1) \cap B\Big) \cup \Big(((V^- \cap B) + i\tilde{Z}_2) \cap B\Big) \supset \\ &\supset \Big((N \cap B) + i(\Gamma + \tilde{Z}_1)') \cap B\Big) \cup \Big((N \cap B) + i(-\Gamma + \tilde{Z}_2)') \cap B\Big)\end{aligned}$$

where " $'$ " means any proper subcone. But $(\Gamma + \tilde{Z}_1)'$ (resp. $(-\Gamma + \tilde{Z}_2)'$) is a conic neighborhood of Γ' (resp. $-\Gamma'$). It follows that $\text{c.h.}(\Gamma + \tilde{Z}_1)' \cup (-\Gamma + \tilde{Z}_2)') = \mathbb{R}^l$ and the conclusion follows, e.g., from the Edge of the Wedge Theorem by Ayrapetian–Henkin. ∎

Remark 6: When M is a hypersurface and N is totally real, Corollary 5 is contained in [6, Th. 1.2]. In fact, our assumptions imply that there exists a vector in Γ which is isotropic for the Levi form of M (in addition to the couple of vectors of opposite sign).

We shall consider analytic discs in $\mathbb{C}^n$, $C^{3,\beta}$ up to the boundary $(0 < \beta < 1)$. These are described by a map $A: \Delta \to \mathbb{C}^n$, $\tau \mapsto A(\tau)$ with A holomorphic in Δ and $C^{3,\beta}$ in $\bar\Delta$ (here $\tau = re^{i\theta}$ is the variable in the standard disc $\bar\Delta$). We shall denote by A both the discs and their parametrizations.

Proof of Theorem 2: We assume $ib = ie_{m+1}$, $c = e_{l+1}$; our hypothesis is therefore

$$(\partial^2_{\bar{z}_{m+1}z_{m+1}}g_j + \partial^2_{\bar{w}_{l+1}w_{l+1}}g_j - 2\Im\mathrm{m}\partial^2_{z_{m+1}\bar{w}_{l+1}}g_j) \neq 0 \quad \text{for some } j.$$

We choose coordinates such that the vector with the above components is the first unit vector e_1 in the $\mathbb{R}^m$-plane. We attach a family of discs A to $V \cup N$ with t components $t(\tau) = \eta\Re\mathrm{e}(1-\tau)e_{m+1}$ and w components $w(\tau) = \eta(1-\tau)e_{l+1}$. For this purpose we solve the Bishop equation

$$u_i(\tau) = -T_1 h_i(u(\tau), t(\tau), w(\tau)) \text{ for } i = 1, \ldots, l, \quad \tau \in \partial\Delta \tag{11}$$

where T_1 is the Hilbert transform normalized by the condition $T_1 u(1) = 0$. Since the h_i's are C^5, it is well known that this can be solved in the Banach space $C^{3,\beta}$, for small data $t(\tau)$ and $w(\tau)$, by the aid of the implicit function theorem (cf. [4]). If we set $z = u + iv$ with $v = T_1 u$ and $A = (z; w)$, then A extends holomorphically from $\partial\Delta$ to Δ and verifies $\partial A \subset N \cup V$ due to $v|_{\partial\Delta} = h(u,t,w)|_{\partial\Delta}$ with $t \in \Gamma$. Note that $A = \{z_o\}$ for $\eta = 0$ and hence $\partial_\tau A = 0$ for $\eta = 0$. Also, it is easy to check that A is $C^{3,\alpha}$ $\forall\alpha < \beta$, in both τ and η. (The shrinking from β to α depends on the fact that T_1 does not preserve $C^{k,\beta}$ smoothness in the parameters; cf. [7].) In particular, $\partial_\tau A$ is C^2 in η. We set $\tau = re^{i\theta}$ and consider the Taylor expansion of $\partial_r A$ in η:

$$\partial_r A = (\partial_r\partial_\eta A|_{\eta=0})\,\eta + \left(\partial_r\partial^2_{\eta\eta}A|_{\eta=0}\right)\frac{\eta^2}{2} + o(\eta^2). \tag{12}$$

We want to prove that for suitably small η the projection of $\partial_r A$ in the $\mathrm{i}\mathbb{R}^m_{y''}$-plane is close to $-2\mathrm{i}e_1$; in particular, A is transversal to M. By (2) and (3) and since $A|_{\eta=0} = \{z_o\}$, when $\eta = 0$ we have $\partial_{x_j}h_i = \partial_{w_k}h_i = 0\,\forall i,j,k$, $\partial_{t_h}h_i = \delta_{hi}$ for $h, i = m+1, \ldots, l$. In particular, differentiation of $v(\tau) = h(u(\tau), t(\tau), w(\tau))$ along $\partial\Delta$ and evaluation for $\eta = 0$ yields

$$\partial_\eta v_i = 0 \quad \forall i \neq m+1\ \forall\tau \in \Delta, \quad \partial_\eta v_{m+1} = \Re\mathrm{e}(1-\tau) \tag{13}$$

(due to $\partial_{t_{m+1}}h_{m+1}(0) = 1$). By differentiating in η the identity $u = -T_1 v$ and interchanging ∂_η with T_1, we get

$$\partial_\eta u_i = 0 \quad \forall i \neq m+1. \tag{14}$$

It follows that

$$\partial_r\partial_\eta(u_i + \mathrm{i}v_i)|_{\eta=0} = 0 \quad \forall i \neq m+1. \tag{15}$$

We prove now that

$$(16)\qquad \left(\partial_r \partial^2_{\eta\eta} v_i\right)_{i=1,\dots,m} = -2e_1.$$

To see this we solve $t = t(x, y'', w)$ on M and write, as in the initial remark, $g_i(x', z'', w) := h_i(x, t(x, y'', w), w)$, $i = 1, \dots, m$. Recall that according to Proposition 1, we can choose g which coincides with its Levi form apart from an error of order higher than 2. Since the discs A are attached to $V \cup N$, whence to M, then $v_i = g_i$ must hold. Double differentiation in η yields

$$(17)\qquad \partial^2_{\eta\eta} v_i|_{\eta=0} = \sum_{m+1\le p,q\le l<h,k} \Big(\partial^2_{z_p\bar z_q} g_i \partial_\eta z_p \partial_\eta \bar z_q + 2\Re(\partial^2_{w_k\bar z_q} g_i \partial_\eta \bar z_q \partial_\eta w_k) + \partial^2_{w_h\bar w_k} g_i \partial_\eta w_h \partial_\eta \bar w_k|_{\eta=0}\Big).$$

Since $\partial_\eta z_p|_{\eta=0} = 0\,\forall p \neq m+1$ and $\partial_\eta w_k|_{\eta=0} = 0\ \forall k \neq l+1$, we get

$$\partial^2_{\eta\eta} v_i|_{\eta=0} = \partial^2_{z_{m+1}\bar z_{m+1}} g_i |\partial_\eta z_{m+1}|^2 + 2\Re\partial^2_{z_{m+1}\bar w_{l+1}} g_i \partial_\eta z_{m+1} \partial_\eta \bar w_{l+1} + \partial^2_{w_{l+1}\bar w_{l+1}} g_i |\partial_\eta w_{l+1}|^2.$$

On the other hand, we have $\partial_\eta z_{m+1}|_{\eta=0} = \partial_\eta u_{m+1} + \mathrm{i}\partial_\eta t_{m+1}\big|_{\eta=0} = \mathrm{i}(1-\tau)$ due to (13), and we also have $\partial_\eta w_{l+1}|_{\eta=0} = 1-\tau$. It follows that

$$(18)\qquad \left(\bar\partial\partial g_i(\overline{ie_{m+1}+e_{l+1}}, ie_{m+1}+e_{l+1})|1-\tau|^2\right)_{i=1,\dots,m} = (|1-\tau|^2)e_1.$$

Note that $(|1-\tau|^2|_{\partial\Delta}) = (2-2\cos(\theta)) = 2(\Re(1-\tau)|_{\partial\Delta})$. Hence by combining (17) and (18) we get

$$(19)\qquad \partial^2_{\eta\eta} v_i = 2e_1 \Re(1-\tau) \quad \text{(all over } \Delta).$$

Differentiation of (19) in r yields (16). Hence A points to $-\partial_r A$ whose normal projection to M is $\mathrm{i}e_1$. We go back to (12), recall (15), and conclude that

$$(20)\qquad \mathrm{pr}(\partial_r v) = -e_1\eta^2 + o(\eta^2).$$

We produce now the manifold V_1 in the statement. We denote as always by $z(\tau) = u(\tau) + \mathrm{i}v(\tau)$, resp. $w(\tau)$, resp. $t(\tau)$ the z, resp. w, resp. t components of A; they are related by $v_j(\tau) = h_j(u(\tau), t(\tau), w(\tau))$, $1 \le j \le l$. For any $s_o \in \mathbb{R}^l_x$, $w_o \in \mathbb{C}^{n-l}_w$, and $t_o \in \Gamma$, we solve the Bishop equation

$$(21)\qquad u_i = -T_1 h_i(u, t(\tau) + t_o, w(\tau) + w_o) + s_o.$$

This produces a family of discs $A = A_{\eta_o s_o t_o w_o}(\tau)$ such that, for any fixed η_o, the map

$$\begin{array}{cccc} D: & \{1-\varepsilon < r < 1\} \times \mathbb{R}^l_{x'} \times \Gamma_t \times \mathbb{C}^{n-l}_w & \to & \mathbb{C}^n \\ & (r, s_o, t_o, w_o) & \mapsto & A_{\eta_o s_o t_o w_o}(r) \end{array}$$

has real rank $2n - m + 1$ (because the A's are transversal to M with a uniform estimate from below for the angle between M and A when $\tau = 1$). We define V_1 to be the range of D in a neighborhood of $z_o = 0$ for all small η_o. It is clear that V_1 is a wedge such that $V \subset \bar{V}_1$ and which points to the directions of the vector field $\mathcal{V} := -\partial_\tau D|_{\tau=1}$. Also, by (20), for all ε and for suitable η, we can arrange that $|\mathrm{pr}(\mathcal{V}(z_o)) - e_1| < \varepsilon$.

We are ready to conclude. By a slight variant of the Baouendi–Treves's Approximation Theorem, there is a neighborhood B of z_o such that any CR function f on $V \cap B$ is uniformly approximated by polynomials over any compact subset. To prove this, one chooses a totally real maximal submanifold of N, e.g., $\{y = h(x, 0, \Re ew)\}$, "pushes it inward V" by choosing $t_1 \in \Gamma$ (small) and defining $S = \{y = h(x, t_1, \Re ew)\}$, and finally takes convolution of f with the heat kernel along S. Next, we take any η_o, s_o, t_o, w_o but require in addition that they are so small that $\partial A_{\eta_o s_o t_o w_o} \subset (V \cup N) \cap B$. But then, for any ϵ, for any $\Gamma' \subset\subset \Gamma$ and for a suitable compact subset $K_{\varepsilon,\Gamma'} \subset\subset V \cap B$, we have

$$\partial A_{\eta_o s_o t_o w_o} \subset K_{\varepsilon,\Gamma'} \quad \text{if } |t_o| \geq \varepsilon,\ t_o \in \Gamma'.$$

Then by the maximum principle on the discs $A_{\eta_o s_o t_o w_o}$, the sequence of polynomials which approximate f in $K_{\varepsilon\Gamma'}$ will converge in the whole discs $A_{\eta_o s_o t_o w_o}$ and will produce the extension of f to V_1. ∎

Proof of Corollary 4: We choose $t_1 \in \Gamma$ and replace V by

$$V_\delta := \{y = h(x, \delta t_1 + t, w)\ t \in \Gamma\}$$

(with $V_\delta \cap B \subset\subset V$, since Γ is open and convex). Thus f is now continuous up to N_δ, the edge of V_δ. Next, we take a polyhedral approximation of Z_Γ that is a family of vectors $a_i \in \Gamma$ such that the cone engendered by the a_i's is proper in Z_Γ. By Theorem 2, f extends to a family of wedges $V_{\delta i}$ which point to additional directions $\tilde{a}_i$ with $|\mathrm{pr}(\tilde{a}_i) - a_i| < \varepsilon$ for any i; moreover, the transversality of the $\tilde{a}_i$ to M is uniform with respect to δ. Since f is continuous in N_δ, we can apply the edge of the Wedge Theorem and obtain that f extends to a wedge V_δ which points to all the directions of $\tilde{Z}' \subset\subset \mathrm{c.h.}\{\mathbb{R}^+ a_i\}$. To conclude, we just let $\delta \to 0$. ∎

Remark 7: The above procedure of "pushing" V along δt_1 in order to get continuity of f in the edge has already been used by Tumanov in [8, Remark 2.5 and the proof of Corollary 2.7]. It is essential to remark that the approximation theorem and the construction of analytic discs are stable under small perturbations, such as pushing V along $t_1 \delta$.

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